Entropy and Growth Rate of Periodic Points of Algebraic $\mathbb{Z}^d$-actions

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Abstract. Expansive algebraic $\mathbb{Z}^d$-actions corresponding to ideals are characterized by the property that the complex variety of the ideal is disjoint from the multiplicative unit torus. For such actions it is known that the limit for the growth rate of periodic points exists and equals the entropy of the action. We extend this result to actions for which the complex variety intersects the multiplicative torus in a finite set. The main technical tool is the use of homoclinic points which decay rapidly enough to be summable.

1. Introduction

An *algebraic* $\mathbb{Z}^d$*-action* on a compact abelian group X is a homomorphism $\alpha\colon \mathbb{Z}^d \to \operatorname{aut}(X)$ from $\mathbb{Z}^d$ to the group of (continuous) automorphisms of X. We denote the image of $\mathbf{n} \in \mathbb{Z}^d$ under α by $\alpha^{\mathbf{n}}$, so that $\alpha^{\mathbf{m}+\mathbf{n}} = \alpha^{\mathbf{m}} \circ \alpha^{\mathbf{n}}$ and $\alpha^{\mathbf{0}} = \mathrm{Id}_X$.

We will consider here cyclic algebraic $\mathbb{Z}^d$-actions, described as follows. Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ denote the ring of Laurent polynomials with integer coefficients in the variables $u_1, \dots, u_d$. We write $f \in R_d$ as $f = \sum_{\mathbf{m}\in\mathbb{Z}^d} f_{\mathbf{m}} \mathbf{u}^{\mathbf{m}}$, where $\mathbf{u} = (u_1, \dots, u_d)$, $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, $\mathbf{u}^{\mathbf{m}} = u_1^{m_1} \dots u_d^{m_d}$, and $f_{\mathbf{m}} \in \mathbb{Z}$ with $f_{\mathbf{m}} = 0$ for all but finitely many $\mathbf{m}$.

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and define the shift $\mathbb{Z}^d$-action σ on $\mathbb{T}^{\mathbb{Z}^d}$ by

$$(\sigma^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d}$. For $f = \sum f_{\mathbf{m}} \mathbf{u}^{\mathbf{m}} \in R_d$ put

$$f(\sigma) = \sum_{\mathbf{m}\in\mathbb{Z}^d} f_{\mathbf{m}} \sigma^{\mathbf{m}} \colon \mathbb{T}^{\mathbb{Z}^d} \to \mathbb{T}^{\mathbb{Z}^d}.$$

We identify R_d with the dual group of $\mathbb{T}^{\mathbb{Z}^d}$ by setting

$$\langle f, x\rangle = e^{2\pi i [f(\sigma)x]_{\mathbf{0}}} = e^{2\pi i \sum_{\mathbf{m}} f_{\mathbf{m}} x_{\mathbf{m}}}$$

for $f \in R_d$ and $x \in \mathbb{T}^{\mathbb{Z}^d}$. In this identification the shift $\sigma^{\mathbf{m}}$ is dual to multiplication by $\mathbf{u}^{\mathbf{m}}$ on R_d.

A closed subgroup $X \subset \mathbb{T}^{\mathbb{Z}^d}$ is shift-invariant if and only if its annihilator

$$X^{\perp} = \{h \in R_d : \langle h, x\rangle = 1 \text{ for every } x \in X\}$$

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is an ideal in R_d. In view of this, for every ideal $\mathfrak{a}$ in R_d write

$$X_{R_d/\mathfrak{a}} = \mathfrak{a}^\perp = \{x \in \mathbb{T}^{\mathbb{Z}^d} : \langle h, x\rangle = 1 \text{ for every } h \in \mathfrak{a}\}$$

for the closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$ annihilated by $\mathfrak{a}$. Note that the dual group of $X_{R_d/\mathfrak{a}}$ is $R_d/\mathfrak{a}$. Denote by $\alpha_{R_d/\mathfrak{a}}$ the restriction of the shift-action σ on $\mathbb{T}^{\mathbb{Z}^d}$ to $X_{R_d/\mathfrak{a}}$. A *cyclic algebraic $\mathbb{Z}^d$-action* is one of this form, corresponding to the cyclic R_d-module $R_d/\mathfrak{a}$.

According to [**9**, Eqn. (1-1)] or [**13**, Thm. 18.1], the topological entropy of $\alpha_{R_d/\mathfrak{a}}$, which coincides with its entropy with respect to Haar measure on $X_{R_d/\mathfrak{a}}$, is given by

$$\mathsf{h}(\alpha_{R_d/\mathfrak{a}}) = \begin{cases} \infty & \text{if } \mathfrak{a} = \{0\}, \\ \mathsf{m}(f) & \text{if } \mathfrak{a} = \langle f\rangle = f \cdot R_d \text{ for some nonzero } f \in R_d, \\ 0 & \text{if } \mathfrak{a} \text{ is nonprincipal}, \end{cases} \tag{1.1}$$

where

$$\mathsf{m}(f) = \int_0^1 \cdots \int_0^1 \log|f(e^{2\pi i t_1}, \dots, e^{2\pi i t_d})|\, dt_1 \dots dt_d$$

is the *logarithmic Mahler measure* of f.

An algebraic $\mathbb{Z}^d$-action α on X is *expansive* if there is a neighborhood U of 0_X such that $\bigcap_{\mathbf{m}\in\mathbb{Z}^d} \alpha^{\mathbf{m}}(U) = \{0_X\}$. To characterize expansiveness for cyclic actions $\alpha_{R_d/\mathfrak{a}}$, let

$$\mathsf{V}(\mathfrak{a}) = \{(z_1, \dots, z_d) \in (\mathbb{C}^\times)^d : g(z_1, \dots, z_d) = 0 \text{ for all } g \in \mathfrak{a}\}$$

denote its complex variety. Let $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$, so that $\mathbb{S}^d$ is the unit multiplicative d-torus in $(\mathbb{C}^\times)^d$. Define the *unitary variety* of $\mathfrak{a}$ as

$$\mathsf{U}(\mathfrak{a}) = \mathsf{V}(\mathfrak{a}) \cap \mathbb{S}^d = \{(z_1, \dots, z_d) \in \mathsf{V}(\mathfrak{a}) : |z_1| = \dots = |z_d| = 1\}.$$

According to [**13**, Thm. 6.5], $\alpha_{R_d/\mathfrak{a}}$ is expansive if and only if $\mathsf{U}(\mathfrak{a}) = \varnothing$.

In order to describe periodic points for $\alpha_{R_d/\langle f\rangle}$, let $\mathcal{F}$ denote the collection of finite-index subgroups of $\mathbb{Z}^d$, and let Γ be an arbitrary element of $\mathcal{F}$. Define $\langle\Gamma\rangle = \min\{\|\mathbf{m}\| : \mathbf{0} \neq \mathbf{m} \in \Gamma\}$, where $\|\mathbf{m}\| = \max\{|m_1|, \dots, |m_d|\}$. A point $x \in X$ has *period* Γ if $\alpha^{\mathbf{m}} x = x$ for all $\mathbf{m} \in \Gamma$. Let

$$\mathsf{Fix}_\Gamma(\alpha_{R_d/\mathfrak{a}}) = \{x \in X_{R_d/\mathfrak{a}} : x \text{ has period } \Gamma\}$$

be the closed subgroup of $X_{R_d/\mathfrak{a}}$ consisting of all Γ-periodic points. In general $\mathsf{Fix}_\Gamma(\alpha_{R_d/\mathfrak{a}})$ may be infinite (examples are given in the next section). We can, however, reduce this to a finite object by forming the quotient of $\mathsf{Fix}_\Gamma(\alpha_{R_d/\mathfrak{a}})$ by its connected component $\mathsf{Fix}_\Gamma^\circ(\alpha_{R_d/\mathfrak{a}})$ of the identity. We therefore define

$$\mathsf{P}_\Gamma(\alpha_{R_d/\mathfrak{a}}) = |\mathsf{Fix}_\Gamma(\alpha_{R_d/\mathfrak{a}}) / \mathsf{Fix}_\Gamma^\circ(\alpha_{R_d/\mathfrak{a}})|,$$

where $|\cdot|$ denotes cardinality. The growth rate of periodic components is defined as

$$\mathsf{p}^+(\alpha_{R_d/\mathfrak{a}}) = \limsup_{\langle\Gamma\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Gamma|} \log \mathsf{P}_\Gamma(\alpha_{R_d/\mathfrak{a}}). \tag{1.2}$$

The following relation between entropy and growth rate of P_Γ was proved in [**13**, Thm. 21.1].

Theorem 1.1. *Let* $\mathfrak{a}$ *be a nonzero ideal in* R_d. *Then* $\mathsf{p}^+(\alpha_{R_d/\mathfrak{a}}) = \mathsf{h}(\alpha_{R_d/\mathfrak{a}})$. *If* $\alpha_{R_d/\mathfrak{a}}$ *is expansive, or equivalently if* $\mathsf{U}(\mathfrak{a}) = \varnothing$, *then the* $\limsup$ *in* (1.2) *is actually a limit, i.e.,*

$$\lim_{\langle\Gamma\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Gamma|} \log \mathsf{P}_\Gamma(\alpha_{R_d/\mathfrak{a}}) = \mathsf{h}(\alpha_{R_d/\mathfrak{a}}). \tag{1.3}$$

It is not known whether (1.3) holds for all cyclic actions. Even when $d = 1$ the existence of this limit involves some deep results in number theory (see [**7**, Sec. 4] for details). The purpose of this note is to prove the following partial result.

Theorem 1.2. *Let* $d \geqslant 2$ *and* $\mathfrak{a}$ *be an ideal in* R_d *whose unitary variety* $\mathsf{U}(\mathfrak{a})$ *finite. Then* (1.3) *holds.*

The machinery described in [**13**, Sec. 21] allows us to reduce the proof of Theorem 1.2 to the case where the ideal $\mathfrak{a}$ is prime. If a prime ideal $\mathfrak{a}$ is nonprincipal, then by (1.1) and Theorem 1.1, $\mathsf{p}^+(\alpha_{R_d/\mathfrak{a}}) = \mathsf{h}(\alpha_{R_d/\mathfrak{a}}) = 0$, which implies (1.3). In view of this fact, we can assume from now on that $\mathfrak{a} = \langle f\rangle$ for some nonzero irreducible Laurent polynomial $f \in R_d$ with

$$|\mathsf{U}(f)| := |\mathsf{U}(\langle f\rangle)| = |\{\mathbf{s} \in \mathbb{S}^d : f(\mathbf{s}) = 0\}| < \infty.$$

Furthermore we will assume for the remainder of this paper that $d \geqslant 2$, for reasons which we will explain. In order to simplify notation, we use X for $X_{R_d/\langle f\rangle}$ and α for $\alpha_{R_d/\langle f\rangle}$.

2. Counting periodic components

In this section we derive an expression for $\mathsf{P}_\Gamma(\alpha) = \mathsf{P}_\Gamma(\alpha_{R_d/\langle f\rangle})$ in terms of f.

Let $\Gamma \in \mathcal{F}$. Following [**13**, (21.13)], we set $\mathfrak{b}_\Gamma = \langle \mathbf{u^m} - 1 : \mathbf{m} \in \Gamma\rangle \subset R_d$, and

$$\Omega_\Gamma = \{\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in \mathbb{S}^d : \boldsymbol{\omega}^{\mathbf{m}} = \omega_1^{m_1} \cdots \omega_d^{m_d} = 1 \text{ for every } \mathbf{m} \in \Gamma\}.$$

Observe that $\Omega_\Gamma = \mathsf{U}(\mathfrak{b}_\Gamma)$. As in [**13**, Sec. 21], we note that the dual group of $\mathsf{Fix}_\Gamma(\alpha) = \mathsf{Fix}_\Gamma(\alpha_{R_d/\langle f\rangle})$ is

$$\mathsf{Fix}_\Gamma(\alpha_{R_d/\langle f\rangle})\widehat{\ } = R_d/(\langle f\rangle + \mathfrak{b}_\Gamma).$$

Hence $\mathsf{P}_\Gamma(\alpha)$ is the cardinality of the $\mathbb{Z}$-torsion subgroup of $R_d/(\langle f\rangle + \mathfrak{b}_\Gamma)$. The following result shows how to compute this number.

Lemma 2.1. *For every finite-index subgroup* $\Gamma \subset \mathbb{Z}^d$,

$$\mathsf{P}_\Gamma(\alpha) = \mathsf{P}_\Gamma(\alpha_{R_d/\langle f\rangle}) = \prod_{\boldsymbol{\omega}\in\Omega_\Gamma\smallsetminus\mathsf{U}(f)} |f(\boldsymbol{\omega})|.$$

Proof. The group $R_d/\mathfrak{b}_\Gamma$ is dual to the group $\mathsf{Fix}_\Gamma(\sigma)$ of Γ-periodic points in $\mathbb{T}^{\mathbb{Z}^d}$. Furthermore, $\mathsf{Fix}_\Gamma(\sigma)$ is isomorphic to the finite-dimensional torus $\mathbb{T}^{\mathbb{Z}^d/\Gamma}$. Then $\mathsf{Fix}_\Gamma(\alpha)$ is the kernel of the restriction of the homomorphism $f(\sigma)$ to this torus $\mathsf{Fix}_\Gamma(\sigma)$.

To describe this kernel we write $\ell^\infty(\mathbb{Z}^d, \mathbb{C})$, $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$, and $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ for the spaces of bounded complex, real, and integer valued functions on $\mathbb{Z}^d$. Let $\tilde{\sigma}$ be the natural shift-action on each of these spaces. Write $V_\Gamma(\mathbb{C}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{C})$, $V_\Gamma(\mathbb{R}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$, and $V_\Gamma(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ for the subspaces of Γ-invariant elements in these spaces.

Next we diagonalize the restriction of $\tilde{\sigma}$ to $V_\Gamma(\mathbb{C})$. For each $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in \Omega_\Gamma$ define $v^{(\boldsymbol{\omega})} \in V_\Gamma(\mathbb{C})$ by

$$(v^{(\boldsymbol{\omega})})_{\mathbf{n}} := \boldsymbol{\omega}^{\mathbf{n}} = \omega_1^{n_1} \cdots \omega_d^{n_d} \quad \text{for } \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d. \tag{2.1}$$

Then $\tilde{\sigma}^{\mathbf{m}} v^{(\boldsymbol{\omega})} = \boldsymbol{\omega}^{\mathbf{m}} v^{(\boldsymbol{\omega})}$ for all $\mathbf{m} \in \mathbb{Z}^d$, and so $f(\tilde{\sigma}) v^{(\boldsymbol{\omega})} = f(\boldsymbol{\omega}) v^{(\boldsymbol{\omega})}$. The set $\{v^{(\boldsymbol{\omega})} : \boldsymbol{\omega} \in \Omega_\Gamma\}$ forms a basis of $V_\Gamma(\mathbb{C})$ consisting of eigenvectors of $\tilde{\sigma}$ with distinct eigenvalues.

Let $\Omega_\Gamma' = \{\boldsymbol{\omega} \in \Omega_\Gamma : f(\boldsymbol{\omega}) \neq 0\} = \Omega_\Gamma \smallsetminus \mathsf{U}(f)$, and define $V_\Gamma'(\mathbb{C})$ to be the $\mathbb{C}$-linear span of $\{v^{(\boldsymbol{\omega})} : \boldsymbol{\omega} \in \Omega_\Gamma'\}$. Then $V_\Gamma'(\mathbb{C})$ is finite-dimensional and $f(\tilde{\sigma})$-invariant. Observe that $f(\tilde{\sigma})(V_\Gamma(\mathbb{C})) = V_\Gamma'(\mathbb{C})$ and that the restriction of $f(\tilde{\sigma})$ to $V_\Gamma'(\mathbb{C})$ is invertible with

$$|\det(f(\tilde{\sigma})|_{V_\Gamma'(\mathbb{C})})| = \prod_{\boldsymbol{\omega} \in \Omega_\Gamma'} |f(\boldsymbol{\omega})|^2.$$

Since $f(\tilde{\sigma})$ commutes with complex conjugation on $V_\Gamma'(\mathbb{C})$, we can restrict it to $V_\Gamma'(\mathbb{R}) = V_\Gamma'(\mathbb{C}) \cap \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ and obtain that

$$|\det(f(\tilde{\sigma})|_{V_\Gamma'(\mathbb{R})})| = \prod_{\boldsymbol{\omega} \in \Omega_\Gamma'} |f(\boldsymbol{\omega})|.$$

The space $V_\Gamma'(\mathbb{Z}) = V_\Gamma'(\mathbb{R}) \cap \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ is a $\tilde{\sigma}$-invariant lattice in $V_\Gamma'(\mathbb{R})$, hence $f(\tilde{\sigma})$-invariant with image $f(\tilde{\sigma})(V_\Gamma'(\mathbb{Z})) \subset V_\Gamma'(\mathbb{Z})$. It follows that

$$|V_\Gamma'(\mathbb{Z}) / f(\tilde{\sigma})(V_\Gamma'(\mathbb{Z}))| = \prod_{\boldsymbol{\omega} \in \Omega_\Gamma'} |f(\boldsymbol{\omega})|. \tag{2.2}$$

Finally, we note that $V_\Gamma(\mathbb{Z}) \cong \mathbb{Z}^{\mathbb{Z}^d/\Gamma}$ is (isomorphic to) the dual group of $\mathsf{Fix}_\Gamma(\sigma) \subset \mathbb{T}^{\mathbb{Z}^d}$, that $V_\Gamma(\mathbb{Z})/f(\tilde{\sigma})(V_\Gamma(\mathbb{Z})) = V_\Gamma(\mathbb{Z})/f(\tilde{\sigma})(V_\Gamma'(\mathbb{Z}))$ is dual to $\mathsf{Fix}_\Gamma(\alpha)$, and that the torsion subgroup $V_\Gamma'(\mathbb{Z})/f(\tilde{\sigma})(V_\Gamma'(\mathbb{Z}))$ of $V_\Gamma(\mathbb{Z})/f(\tilde{\sigma})(V_\Gamma'(\mathbb{Z}))$ is therefore dual to $\mathsf{Fix}_\Gamma(\alpha)/\mathsf{Fix}_\Gamma^\circ(\alpha)$. By combining this with (2.2) we complete the proof. $\square$

Remark 2.2. Suppose that $\alpha_{R_d/\langle f \rangle}$ is expansive, so that $\mathsf{U}(f) = \varnothing$. Then f does not vanish on $\mathbb{S}^d$, so $\log|f|$ is continuous there. Lemma 2.1 shows that

$$\frac{1}{|\mathbb{Z}^d/\Gamma|} \log \mathsf{P}_\Gamma(\alpha_{R_d/\langle f \rangle}) = \frac{1}{|\mathbb{Z}^d/\Gamma|} \sum_{\boldsymbol{\omega} \in \Omega_\Gamma} \log|f(\boldsymbol{\omega})|$$

is a Riemann sum approximation to $\mathsf{m}(f)$, and so converges to $\mathsf{m}(f) = \mathsf{h}(\alpha)$ as $\langle \Gamma \rangle \to \infty$.

When $\mathsf{U}(f) \neq \varnothing$ there are two issues to deal with. The vanishing of f at some points of Ω_Γ creates connected components, so we count those. More difficult are various diophantine problems concerning points of $\mathsf{U}(f)$ coming abnormally close to Ω_Γ. The latter issue is discussed in Section 9.

3. Examples

We provide here some examples of irreducible polynomials f with finite $\mathsf{U}(f)$, illustrating a range of algebraic properties of $\mathsf{U}(f)$ and the resulting influence on the structure of $\mathsf{Fix}_\Gamma(\alpha_{R_d/\langle f \rangle})$. For clarity we use variables u, v, w, rather than u_1, u_2, u_3.

Example 3.1. Let $d = 2$ and $f(u,v) = 2 - u - v$. Clearly $\mathsf{U}(f) = \{(1,1)\}$. Observe that $F = \mathsf{Fix}_{\mathbb{Z}^2}(\alpha_{R_2/\langle 2-u-v\rangle})$ is isomorphic to $\mathbb{T}$, with each $t \in \mathbb{T}$ corresponding to the point in $X_{R_2/\langle 2-u-v\rangle}$ all of whose coordinates equal t. For each finite-index subgroup Γ we have that $\Omega_\Gamma \cap \mathsf{U}(f) = \{(1,1)\}$, so the analysis of the previous section implies that $\mathsf{Fix}_\Gamma(\alpha_{R_2/\langle 2-u-v\rangle})$ is a finite union of cosets of F, and hence always infinite. The exact number of these cosets is computed in Lemma 2.1.

Example 3.2. Let $d = 2$ and $f(u,v) = 1 + u + v$. Setting $\omega = e^{2\pi i/3}$, it is easy to verify that $\mathsf{U}(f) = \{(\omega,\omega^2),(\omega^2,\omega)\}$. To describe the periodic point behavior of this example, parametrize the finite-index subgroups of $\mathbb{Z}^2$ as

$$\Gamma_{a,b,c} = \mathbb{Z}\begin{bmatrix} a \\ 0 \end{bmatrix} \oplus \mathbb{Z}\begin{bmatrix} b \\ c \end{bmatrix}, \text{ where } a > 0,\ c > 0, \text{ and } 0 \leqslant b < a.$$

Then

$$\Omega_{\Gamma_{a,b,c}} \cap \mathsf{U}(f) = \begin{cases} \mathsf{U}(f) & \text{if } a \equiv 0 \pmod 3 \text{ and } b + 2c \equiv 0 \pmod 3, \\ \varnothing & \text{otherwise.} \end{cases}$$

Hence $\mathsf{Fix}_{\Gamma_{a,b,c}}(\alpha_{R_2/\langle 1+u+v\rangle})$ is a finite union of 2-dimensional tori if $a \equiv 0 \pmod 3$ and $b + 2c \equiv 0 \pmod 3$, and is a finite set otherwise. Thus $\mathsf{Fix}_{\Gamma_{N\mathbb{Z}^2}}(\alpha_{R_2/\ 1+u+v\rangle})$ is infinite whenever N is a multiple of 3. In this example the coordinates of every point in $\mathsf{U}(f)$ are roots of unity.

Example 3.3. Let $d = 2$ and $f(u,v) = 2 - u^2 + v - uv$. We will show that $\mathsf{U}(f) = \{(\xi,\eta),(\overline{\xi},\overline{\eta})\}$, where ξ and η are algebraic numbers but not algebraic integers. It follows that $\Omega_\Gamma \cap \mathsf{U}(f) = \varnothing$ for all $\Gamma \in \mathcal{F}$, and hence that $\mathsf{Fix}_\Gamma(\alpha_{R_2/\langle f\rangle})$ is always finite.

From $f(u,v) = 0$ we obtain that $v = v(u) = \frac{2-u^2}{u-1}$. Setting $u = e^{2\pi i\theta}$, we must solve $|v(e^{2\pi i\theta})| = 1$. Since $\overline{u} = u^{-1} = e^{-2\pi i\theta}$, we can write $1 = |v(u)|^2 = v(u)v(\overline{u}) = v(u)v(u^{-1})$ as an algebraic equation. Clearing fractions yields $(2-u^2)(2-u^{-2}) = (u-1)(u^{-1}-1)$. Symmetry in u and u^{-1} means we can write this as an equation in $c = \frac{1}{2}(u+u^{-1}) = \cos 2\pi\theta$, resulting in $8c^2 - 2c - 7 = 0$. This equation has roots $(1-\sqrt{57})/8 \approx -0.818$ and $(1+\sqrt{57})/8 \approx 1.068$. Only the first is a possible value of $\cos 2\pi\theta$, so $\mathrm{Re}(\xi) = (1-\sqrt{57})/8$. There are two choices for $\mathrm{Im}(\xi)$, namely $\pm(1-\mathrm{Re}\,(\xi)^2)^{1/2}$. Using these yield the corresponding values $\eta = v(\xi)$, or explicitly,

$$\xi = \frac{1-\sqrt{57}}{8} + i\left(\frac{3+\sqrt{57}}{32}\right)^{1/2}$$

and

$$\eta = \frac{-1}{56+8\sqrt{57}}\left[34 + 6\sqrt{57} + i\left(11\sqrt{6+2\sqrt{57}} + \sqrt{342+114\sqrt{57}}\right)\right]$$

The minimal polynomial for ξ is $2t^4 - t^3 - 3t^2 - t + 2$ and for η is $2t^4 + 13t^3 + 18t^2 + 13t + 2$, showing that each is an algebraic number but not an algebraic integer.

Example 3.4. Let $d = 2$ and $f(u,v) = 2 - u^3 + v - uv - u^2v$. Here v appears linearly, and the techniques used in the preceding example still work. In this case $v(u) = \frac{2-u^3}{u^2+u-1}$, and the equation $v(u)v(u^{-1}) = 1$ is transformed under the change of variables $c = \frac{1}{2}(u+u^{-1})$ to $16c^3 - 4c^2 - 12c = 0$. The root $c = 1$ yields the point $(1,1) \in \mathsf{U}(f)$. The root $c = 0$ gives $u = \pm i$, with corresponding $v(\pm i) = -\frac{3}{5} \mp i\frac{4}{5}$.

The final root $c = -\frac{3}{4}$ gives $u = -\frac{3}{4} \pm i\frac{\sqrt{7}}{4}$, with corresponding $v = -\frac{528}{704} \pm i\frac{176\sqrt{7}}{704}$. Thus

$$\mathsf{U}(f) = \{(1,1),\ (i,\xi),\ (-i,\overline{\xi}),\ (\eta,\zeta),\ (\overline{\eta},\overline{\zeta})\},$$

where

$$\xi = -\frac{3}{5} - i\frac{4}{5},\quad \eta = -\frac{3}{4} + i\frac{\sqrt{7}}{4},\ \text{and}\ \ \zeta = -\frac{528}{704} + i\frac{176\sqrt{7}}{704}$$

are all algebraic numbers but not algebraic integers.

Note that although f is irreducible, the algebraic properties of the coordinates of points in $\mathsf{U}(f)$ vary considerably.

In the previous two examples we exploited the property that one variable could be expressed as a rational function of the other. In general this function will be algebraic, and calculations much more difficult. An alternative approach is to use Gröbner bases. Let $u_k = x_k + iy_k$ and expand $f(x_1 + iy_1, \dots, x_d + iy_d)$ into real and imaginary parts as $g(x_1, y_1, \dots, x_d, y_d) + ih(x_1, y_1, \dots, x_d, y_d)$, where $g, h \in \mathbb{Z}[x_1, y_1, \dots, x_d, y_d]$. Compute a Gröbner basis for the ideal in $\mathbb{Q}[x_1, y_1, \dots, x_d, y_d]$ generated by g, h, and the polynomials $x_k^2 + y_k^2 - 1$ $(1 \leqslant k \leqslant d)$, say with term order $x_1 \prec y_1 \prec \dots \prec x_d \prec y_d$. If this basis contains a polynomial in x_1 only, we can solve for the real roots and back substitute to obtain all solutions. Carrying this out on Example 3.3, for instance, gives $8x_1^2 - 2x_1 - 7$ in the ideal, the same polynomial (in c) as we arrived at there.

Before the next example, we remark that when $d = 2$ finding examples is relatively easy, since generically we expect the 2-dimensional torus to intersect the (real) 2-dimensional variety in a finite set. This behavior fails for $d \geqslant 3$, and the matter is more delicate since the variety must now intersect the torus tangentially in finitely many places.

Example 3.5. Let $d = 3$ and $f(u,v,w) = g(u) - v - w$, where $g(u) = u^4 - 3u^3 + 3u + 3$.

We claim that the minimum value of $|g|$ on $\mathbb{S}$ is 2, and that this minimum is attained at exactly two algebraic integers η and $\overline{\eta}$ in $\mathbb{S}$. It turns out here that $g(\eta) = 2\overline{\eta}$. Hence

$$\mathsf{U}(f) = \{(\eta, \overline{\eta}, \overline{\eta}), (\overline{\eta}, \eta, \eta)\}.$$

It follows that all periodic point groups are finite.

To verify our claim, use the rational function parametrization $s\colon \mathbb{R} \to \mathbb{S} \smallsetminus \{-i\}$ given by

$$s(t) = \frac{2t}{1+t^2} + i\,\frac{1-t^2}{1+t^2}.$$

(Omitting $-i$ from the range is harmless since $-i$ is far from the location of the minimum.) Then

$$\phi(t) = |g(s(t))|^2 = g(s(t))g(\overline{s(t)}) > 0.$$

Expanding this product and taking the derivative shows, after a lengthy calculation, that

$$\phi'(t) = -\frac{96}{(1+t^2)^5}(t^8 - 7t^7 - 10t^6 + 25t^5 - 25t^3 + 10t^2 + 7t - 1).$$

Evaluating ϕ at the real roots of $\phi'(t) = 0$ shows that the minimum value of ϕ is attained at the two real roots of the irreducible quartic factor $t^4 + t^3 - 2t^2 + t + 1$

of the numerator of $\phi'(t)$, explicitly at

$$\xi = \frac{-1-\sqrt{17}}{4} + \frac{1}{2}\sqrt{\frac{1+\sqrt{17}}{2}}$$

and its real conjugate. We put $\eta = s(\xi) \in \mathbb{S}$. An exact calculation shows that $g(\eta) = 2\overline{\eta}$, verifying our claim.

One variation on this theme is to use $f(u,v,w) = g(u) - v^r - w^s$, which has a more complicated, but still finite, unitary variety.

4. Algebraic points on varieties

In every example from the preceding section, the coordinates of the points in $\mathsf{U}(f)$ are algebraic numbers. Using an argument kindly shown to us by Marius van der Put, we will prove that this is always true. The algebraicity of the coordinates is crucial to our proof of Theorem 1.2.

We begin with a result in real algebraic geometry.

Proposition 4.1. *Let $\mathfrak{q}$ be an ideal in $\mathbb{Q}[t_1,\dots,t_d]$ and define*

$$\mathsf{R}(\mathfrak{q}) := \{(r_1,\dots,r_d) \in \mathbb{R}^d : g(r_1,\dots,r_d) = 0 \textit{ for all } g \in \mathfrak{q}\}.$$

Suppose that $(a_1,\dots,a_d)$ is an isolated point in $\mathsf{R}(\mathfrak{q})$. Then each a_j is an algebraic number.

Proof. Each $\mathbf{a} = (a_1,\dots,a_d) \in \mathsf{R}(\mathfrak{q})$ gives a ring homomorphism

$$\phi_{\mathbf{a}} \colon \mathbb{Q}[t_1,\dots,t_d]/\mathfrak{q} \to K := \mathbb{Q}(a_1,\dots,a_d) \subset \mathbb{R}$$

with $\phi_{\mathbf{a}}(t_j) = a_j$, and every homomorphism $\mathbb{Q}[t_1,\dots,t_d]/\mathfrak{q} \to \mathbb{R}$ comes from a point in $\mathsf{R}(\mathfrak{q})$ this way.

Suppose that $\mathbf{a} = (a_1,\dots,a_d) \in \mathsf{R}(\mathfrak{q})$, and that $K = \mathbb{Q}(a_1,\dots,a_d)$ is not algebraic over $\mathbb{Q}$. Then there are $k \geqslant 1$ algebraically independent elements $b_1,\dots,b_k \in K$ and an element $c \in K$ algebraic over $\mathbb{Q}(b_1,\dots,b_k)$ such that $K = \mathbb{Q}(b_1,\dots,b_k)(c)$. Write the minimal polynomial of c over $\mathbb{Q}(b_1,\dots,b_k)$ as

$$P(b_1,\dots,b_k,T) := T^n + p_{n-1}(b_1,\dots,b_k)T^{n-1} + \dots + p_0(b_1,\dots,b_k)$$

where the $p_j(T_1,\dots,T_k) \in \mathbb{Q}(T_1,\dots,T_k)$ are rational functions. Now $P(b_1,\dots,b_k,T)$ is irreducible over $\mathbb{Q}(b_1,\dots,b_k) \subset \mathbb{R}$, so that c is a simple root. Hence there are $c_1 < c$ and $c_2 > c$ such that $P(b_1,\dots,b_k,c_1)$ and $P(b_1,\dots,b_k,c_2)$ are nonzero and have opposite sign. Therefore if we perturb slightly each b_j to b_j', the new polynomial $P(b_1',\dots,b_k',T) \in \mathbb{R}[T]$ has a root c' very close to c. If we further assume that $\{b_1',\dots,b_k'\}$ is also algebraically independent, then there is a field isomorphism

$$\psi \colon \mathbb{Q}(b_1,\dots,b_k,c) \to \mathbb{Q}(b_1',\dots,b_k',c').$$

Now each a_j is in $K = Q(b_1,\dots,b_k)(c)$ and can thus be written in the form

$$a_j = \sum_{m=0}^{n} q_{mj}(b_1,\dots,b_k)c^m, \text{ where } q_{mj}(T_1,\dots,T_k) \in \mathbb{Q}(T_1,\dots,T_k).$$

Hence if the perturbations are sufficiently small, we see that

$$a_j' := \sum_{m=0}^{n} q_{mj}(b_1',\dots,b_k')(c')^m$$

is very close to a_j for $1 \leqslant j \leqslant d$. Let $\mathbf{a}' = (a'_1, \dots, a'_d)$. Then

$$\phi_{\mathbf{a}'} = \psi \circ \phi_{\mathbf{a}} \colon \mathbb{Q}[t_1, \dots, t_d]/\mathfrak{q} \to \mathbb{R}$$

is a homomorphism, and so $\mathbf{a}' \in \mathsf{R}(\mathfrak{q})$. This proves that if $\mathbf{a}$ has at least one non-algebraic coordinate, then $\mathbf{a}$ cannot be isolated in $\mathsf{R}(\mathfrak{q})$. $\square$

Proposition 4.2. *Let $f \in R_d$ and suppose that $\mathsf{U}(f)$ is finite. Then the coordinates of every point in $\mathsf{U}(f)$ are algebraic numbers.*

Proof. We again use the rational function parametrization $s \colon \mathbb{R} \to \mathbb{S} \smallsetminus \{-i\}$ given by

$$s(t) = \frac{2t}{1+t^2} + i\,\frac{1-t^2}{1+t^2}.$$

Define $\mathbf{s} \colon \mathbb{R}^d \to \mathbb{S}^d$ by $\mathbf{s}(t_1, \dots, t_d) = (s(t_1), \dots, s(t_d))$. We may assume that $\mathsf{U}(f) \subset \mathbf{s}(\mathbb{R}^d)$. For if this fails, we can easily adjust the parametrization to omit a point on $\mathbb{S}$ with rational coordinates that does not appear as a coordinate of any point in the finite set $\mathsf{U}(f)$.

Consider the equation $f(\mathbf{s}(t_1, \dots, t_d)) = 0$. Expanding and multiplying through by $\prod_{k=1}^d (1+t_k)^{n_k} \neq 0$ for suitable n_k, this takes the form

$$g_1(t_1, \dots, t_d) + i\, g_2(t_1, \dots, t_d) = 0,$$

where each $g_j \in \mathbb{Z}[t_1, \dots, t_d]$. Let $\mathfrak{q} = \langle g_1, g_2 \rangle \subset \mathbb{Q}[t_1, \dots, t_d]$. By assumption, $\mathsf{R}(\mathfrak{q})$ is finite, so all of its points are isolated. By the preceding proposition, these points have algebraic coordinates. Each point in $\mathsf{U}(f)$ is the image under $\mathbf{s}$ of a point in $\mathsf{R}(\mathfrak{q})$, and hence also has coordinates that are algebraic numbers. $\square$

5. Homoclinic points

In this section we will construct periodic points by using homoclinic points which decay rapidly enough.

Let β be an algebraic $\mathbb{Z}^d$-action on a compact abelian group Y. An element $y \in Y$ is *homoclinic for* β if $\lim_{|\mathbf{n}| \to \infty} \beta^{\mathbf{n}} y = 0_Y$. The set of all homoclinic points for β is a subgroup of Y, denoted by $\Delta_\beta(Y)$.

According to [**8**], the following hold if β is assumed to be expansive:

(1) $\Delta_\beta(Y)$ is at most countable;
(2) $\Delta_\beta(Y) \neq \{0_Y\}$ if and only if β has positive entropy with respect to Haar measure λ_Y on Y;
(3) $\Delta_\beta(Y)$ is dense in Y if and only if β has completely positive entropy with respect to λ_Y; and
(4) For every $y \in \Delta_\beta(Y)$, $\beta^{\mathbf{n}} y \to 0_Y$ exponentially fast.

If β is not expansive, then there is no guarantee that $\Delta_\beta(Y) \neq \{0\}$, even if β has completely positive entropy. For example, let $A \in GL_n(\mathbb{Z})$ have irreducible characteristic polynomial, and also have some but not all of its roots on $\mathbb{S}$. Then by [**8**, Example 3.4], the $\mathbb{Z}$-action generated by A on $\mathbb{T}^n$ has completely positive entropy (indeed is Bernoulli), and yet has trivial homoclinic group.

Furthermore, if β is not expansive then homoclinic points may decay very slowly, in contrast to the exponential decay in the expansive case. Let $f(u,v) = 2 - u - v$ and consider the $\mathbb{Z}^2$-action $\alpha_{R_2/\langle f \rangle}$ on $X_{R_2/\langle f \rangle}$ that we discussed in

Example 3.1. By [**8**, Example 7.2], $1/\overline{f}$ is integrable on $\mathbb{S}^2$, and its Fourier transform w^Δ is given by

$$w^\Delta_{(-m,-n)} = \begin{cases} \dfrac{1}{2^{m+n+1}}\dbinom{m+n}{n} & \text{if } m \geqslant 0 \text{ and } n \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Let x^Δ denote the coordinate-wise reduction (mod 1) of w^Δ. Then x^Δ is a homoclinic point for $\alpha_{R_2/\langle f\rangle}$, and in fact every homoclinic point is a finite integral combination of translates of x^Δ. Note that

$$w^\Delta_{(-n,-n)} = \frac{1}{2^{2n+1}}\binom{2n}{n} \approx \frac{1}{2\sqrt{\pi n}}$$

decays slowly, and also that $\sum_{m,n}|w^\Delta_{(m,n)}| = \infty$.

When $\mathsf{U}(f)$ is finite but nonempty, the action $\alpha = \alpha_{R_d/\langle f\rangle}$ is not expansive on $X = X_{R_d/\langle f\rangle}$. We will restrict our attention to those homoclinic points which decay rapidly enough to be summable. Hence define

$$\Delta^1_\alpha(X) := \Big\{x \in \Delta_\alpha(X) : \sum_{\mathbf{n}\in\mathbb{Z}^d}|x_\mathbf{n}| < \infty\Big\},$$

where for $t \in \mathbb{T}$ we let $|t|$ denote the distance from t to 0 in $\mathbb{T}$.

In order to analyze the homoclinic group, we first linearize the action α. Consider the surjective map $\rho\colon \ell^\infty(\mathbb{Z}^d,\mathbb{R}) \to \mathbb{T}^{\mathbb{Z}^d}$ given by $\rho(w)_\mathbf{n} = w_\mathbf{n} \pmod 1$. If $f = \sum_\mathbf{n} f_\mathbf{n}\mathbf{u}^\mathbf{n}$ and $\tilde\sigma$ is the shift-action on $\ell^\infty(\mathbb{Z}^d,\mathbb{R})$, then $f(\tilde\sigma) = \sum_\mathbf{n} f_\mathbf{n}\tilde\sigma^\mathbf{n}\colon \ell^\infty(\mathbb{Z}^d,\mathbb{R}) \to \ell^\infty(\mathbb{Z}^d,\mathbb{R})$. We define

$$\begin{aligned} W_f := \rho^{-1}(X) &= \{w \in \ell^\infty(\mathbb{Z}^d,\mathbb{R}) : \rho(w) \in X\} \\ &= \{w \in \ell^\infty(\mathbb{Z}^d,\mathbb{R}) : f(\tilde\sigma)(w) \in \ell^\infty(\mathbb{Z}^d,\mathbb{Z})\}, \end{aligned}$$

and view W_f as the linearization of X.

For $w \in \ell^\infty(\mathbb{Z}^d,\mathbb{R})$, we define its *adjoint* w^* by $w^*_\mathbf{n} = w_{-\mathbf{n}}$. Each $a \in \ell^1(\mathbb{Z}^d,\mathbb{R})$ acts as a linear operator on $\ell^\infty(\mathbb{Z}^d,\mathbb{R})$ via convolution, defined by

$$(a * w)_\mathbf{m} = \sum_{\mathbf{n}\in\mathbb{Z}^d} a_\mathbf{n} w_{\mathbf{m}-\mathbf{n}} \quad \text{for all } w \in \ell^\infty(\mathbb{Z}^d,\mathbb{R}).$$

For $a \in \ell^1(\mathbb{Z}^d,\mathbb{R})$ we define its Fourier transform $\widehat{a}\colon \mathbb{S}^d \to \mathbb{C}$ by $\widehat{a}(\mathbf{s}) = \sum_\mathbf{n} a_\mathbf{n}\mathbf{s}^\mathbf{n}$, where as usual $\mathbf{s}^\mathbf{n} = s_1^{n_1}\cdots s_d^{n_d}$. In the opposite direction, if $\phi\colon \mathbb{S}^d \to \mathbb{C}$ is integrable with respect to Haar measure λ on $\mathbb{S}^d$, then we write $\widetilde{\phi} \in \ell^\infty(\mathbb{Z}^d,\mathbb{C})$ for its Fourier transform, where

$$\widetilde{\phi}_\mathbf{n} = \int_{\mathbb{S}^d}\phi(\mathbf{s})\mathbf{s}^{-\mathbf{n}}\,d\lambda(\mathbf{s}).$$

If $g = \sum_\mathbf{n} g_\mathbf{n}\mathbf{u}^\mathbf{n} \in R_d$, we can consider g as the element $(g_\mathbf{n}) \in \ell^1(\mathbb{Z}^d,\mathbb{R})$. With this convention, the action of $g(\tilde\sigma)$ on $\ell^\infty(\mathbb{Z}^d,\mathbb{R})$ coincides with convolution by g^*, i.e., $g(\tilde\sigma)(w) = g^* * w$. Furthermore, $\widehat{g}$ is just the restriction of the polynomial function g to $\mathbb{S}^d$, and $\widehat{g^*}$ is the restriction of the complex conjugate $\overline{g}$.

Since the Fourier transform $\widehat{f}$ of f has only finitely many zeros on $\mathbb{S}^d$ by assumption, it follows that $1/\widehat{f}\colon \mathbb{S}^d \to \mathbb{C}$ is analytic with finitely many poles. We seek multipliers that will make the Fourier transform summable, and so define

$$\mathfrak{m}_f := \big\{g \in R_d : \widehat{g}/\widehat{f}\colon \mathbb{S}^d \to \mathbb{C} \text{ has absolutely convergent Fourier series}\big\},$$

which is clearly an ideal in R_d. For every $g \in \mathfrak{m}_f$ we write

$$w^{(g)} = (\widehat{g^*}/\widehat{f^*})\breve{\ } \in \ell^1(\mathbb{Z}^d, \mathbb{R}) \tag{5.2}$$

for the summable Fourier transform of $\widehat{g^*}/\widehat{f^*} = \overline{\widehat{g}/\widehat{f}}$.

Proposition 5.1. *Let $f \in R_d$ with finite $\mathsf{U}(f)$. Then $\langle f \rangle \subsetneq \mathfrak{m}_f$.*

Before beginning the proof, we remark that if $\mathsf{U}(f) = \varnothing$, then $1/\widehat{f}$ is smooth, and so $\mathfrak{m}_f = R_d$. However, if $\mathsf{U}(f)$ is nonempty, then $1/\widehat{f}$ is not bounded, and so $\mathfrak{m}_f$ is a proper ideal. The strict containment $\langle f \rangle \subsetneq \mathfrak{m}_f$ fails for $d = 1$, and this is the main reason we require $d \geqslant 2$.

Proof. First note that f cannot be expressed as a polynomial in fewer than d variables since $\mathsf{U}(f)$ is finite. Hence no polynomial of fewer variables can be contained in $\langle f \rangle$.

Define the isomorphism $e\colon \mathbb{T}^d \to \mathbb{S}^d$ by $e(t_1, \dots, t_d) = (e^{2\pi i t_1}, \dots, e^{2\pi i t_d})$. As before, for $t \in \mathbb{T}$ let $|t|$ denote the distance from t to 0. For $\mathbf{t}, \mathbf{t}' \in \mathbb{T}^d$ put $\|\mathbf{t} - \mathbf{t}'\| = \max\{|t_j - t_j'| : 1 \leqslant j \leqslant d\}$. Define the metric δ on $\mathbb{S}^d$ by $\delta(\mathbf{s}, \mathbf{s}') = \|e^{-1}(\mathbf{s}) - e^{-1}(\mathbf{s}')\|$.

Let $\mathbf{a} = (a_1, \dots, a_d) \in \mathsf{U}(f)$. Since $\widehat{f} \circ e$ is analytic on $\mathbb{R}^d$, and in a neighborhood of $e^{-1}(\mathbf{a})$ vanishes only there, it follows that there are constants $c > 0$, $k \geqslant 1$, and $\varepsilon > 0$ such that

$$|\widehat{f}(\mathbf{s})| \geqslant c\,\delta(\mathbf{s}, \mathbf{a})^k \text{ whenever } \delta(\mathbf{s}, \mathbf{a}) < \varepsilon.$$

We start by considering the first coordinate a_1 of $\mathbf{a}$. By Proposition 4.2, a_1 is an algebraic number. Hence there is a nonzero polynomial $h_1 \in \mathbb{Z}[u_1]$ with $h_1(a_1) = 0$. It follows that $|\widehat{h}(s)| \leqslant c_1 \delta_1(s, a_1)$ for $s \in \mathbb{S}$ near a_1, where $c_1 > 0$ is a suitable constant and δ_1 is the metric on $\mathbb{S}$ analogous to δ. Define $h \in R_d$ by $h(u_1, \dots, u_d) = h_1(u_1)$. Then for $\mathbf{s}$ near $\mathbf{a}$ we have that $|\widehat{h}(\mathbf{s})| \leqslant c_1\,\delta_1(s_1, a_1) \leqslant c_1\,\delta(\mathbf{s}, \mathbf{a})$. Hence near $\mathbf{a}$ we have the estimate

$$\left| \frac{\widehat{h}^n(\mathbf{s})}{\widehat{f}(\mathbf{s})} \right| \leqslant \left(\frac{c_1^n}{c} \right) \delta(\mathbf{s}, \mathbf{a})^{n-k}.$$

By taking n sufficiently large we can guarantee that $\widehat{h}^n/\widehat{f}$ is as differentiable as we please, in particular that it is d times continuously differentiable.

Repeating this procedure for every point in $\mathsf{U}(f)$, and letting $g[u_1] \in \mathbb{Z}[u_1]$ be the product of the corresponding $h_1^n(u_1)$'s, we obtain that $\widehat{g}/\widehat{f}$ is d times continuously differentiable on $\mathbb{S}^d$. Hence the Fourier series of $\widehat{g}/\widehat{f}$ is absolutely convergent (see [**10**] or [**1**] for much sharper results). Thus $g \in \mathfrak{m}_f$, and since it is a polynomial in one variable it cannot be in $\langle f \rangle$ by our earlier remark. $\square$

Proposition 5.2. *Suppose that $f \in R_d$ has finite $\mathsf{U}(f)$, and let $\alpha = \alpha_{R_d/\langle f \rangle}$ be the algebraic $\mathbb{Z}^d$-action on $X = X_{R_d/\langle f \rangle}$. For every $g \in \mathfrak{m}_f$ let $x^{(g)} = \rho(w^{(g)})$, where $w^{(g)}$ is defined in* (5.2)*. Then $\Delta^1_\alpha(X) = \{x^{(g)} : g \in \mathfrak{m}_f\}$. Furthermore, $x^{(g)} = 0$ if and only if $g \in \langle f \rangle$, so that the map $g + \langle f \rangle \mapsto x^{(g)}$ is a group isomorphism of $\mathfrak{m}_f/\langle f \rangle$ with $\Delta^1_\alpha(X)$.*

Proof. Let $z \in \Delta^1_\alpha(X)$. Choose $w \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ with $\rho(w) = z$. Then

$$f(\tilde{\sigma})(w) \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \cap \ell^1(\mathbb{Z}^d, \mathbb{R}),$$

and so is an element, say g^*, of R_d. Taking Fourier transforms of $f^* * w = f(\tilde{\sigma})(w) = g^*$ shows that $\widehat{f^*} \cdot \widehat{w} = \widehat{g^*}$. Hence $\widehat{w} = \widehat{g^*}/\widehat{f^*}$ is well-defined off a finite set, and has absolutely convergent Fourier series w. Thus $w = w^{(g)}$, and so $z = x^{(g)}$.

Conversely, suppose that $g \in \mathfrak{m}_f$. Then $w^{(g)} = (\widehat{g^*}/\widehat{f^*})\check{} \in \ell^1(\mathbb{Z}^d, \mathbb{R})$, and as above we obtain that $f(\tilde{\sigma})(w^{(g)}) = g^* \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$. Hence $w^{(g)} \in W_f$, and so $x^{(g)} = \rho(w^{(g)}) \in \Delta^1_\alpha(x)$.

Finally, if $g \in \mathfrak{m}_f$ and $x^{(g)} = \rho(w^{(g)}) = 0$, then $h = w^{(g)} \in R_d$. Taking Fourier transforms gives $\widehat{h} = \widehat{g^*}/\widehat{f^*}$, so that $g^* = h \cdot f^*$ and $g = f \cdot h^* \in \langle f \rangle$. The converse is obvious. □

Sometimes it is useful to determine $\mathfrak{m}_f$ explicitly. For example, this is the case in [**14**], where the Laplacian $f^{(d)} = 2d - \sum_{j=1}^d (u_j + u_j^{-1}) \in R_d$, $d \geqslant 2$, was studied. There it is shown that

$$\mathfrak{m}_{f^{(d)}} = \langle f^{(d)} \rangle + \mathfrak{I}_d^3, \tag{5.3}$$

where $\mathfrak{I}_d = \big\{h \in R_d : h(1, \dots, 1) = 0\big\} = \langle u_1 - 1, \dots, u_d - 1\rangle$.

We demonstrate how to obtain such results using again the example $f(u, v) = 2 - u - v \in R_2$ discussed in Example 3.1 and at the start of Section 5. Firstly, since f has only one zero on $\mathbb{S}^2$, namely $\mathbf{1} = (1, 1)$, the Fourier transform $\widehat{f}$ has one zero on $\mathbb{T}^2 \cong [-1/2, 1/2)^2$ at $\mathbf{0} = (0, 0)$. The Taylor series expansion of $\widehat{f}$ at $\mathbf{0}$ is

$$\widehat{f}(\theta, \phi) = -2\pi i(\theta + \phi) + 2\pi^2(\theta^2 + \phi^2) + \mathcal{O}(|\theta|^3 + |\phi|^3).$$

According to the proof of Proposition 5.1, $g_m(u) := (u-1)^m \in \mathfrak{m}_f$ for all sufficiently large m. What is the minimal such m? If $g_m(u) \in \mathfrak{m}_f$, then $\widehat{g}_m/\widehat{f}$ must be at least continuous at $\mathbf{0}$. Inspecting the Taylor series expansion of $\widehat{g}_m$ at $\mathbf{0}$ for small m we find that

$$\begin{aligned}
\widehat{g}_0(\theta, \phi) &= 1,\\
\widehat{g}_1(\theta, \phi) &= -2\pi i\theta + 2\pi^2\theta^2 + \mathcal{O}(\theta^3),\\
\widehat{g}_2(\theta, \phi) &= -4\pi^2\theta^2 + \mathcal{O}(\theta^3).
\end{aligned}$$

It is evident that $\widehat{g}_m/\widehat{f}$ is not continuous at $\mathbf{0}$ for $m = 0, 1, 2$. It turns out that $g_3(u) \in \mathfrak{m}_f$. We can establish this fact either by showing that $\widehat{g}_3/\widehat{f}$ is sufficiently smooth at $\mathbf{0}$, or alternatively by showing that $g_3^* * w^\Delta \in \ell^1(\mathbb{Z}^d, \mathbb{R})$, where w^Δ is the homoclinic point given by (5.1).

For every $(m, n) \in \mathbb{Z}^2$ we have that

$$(g_3^* * w^\Delta)_{(m,n)} = -w_{(m,n)} + 3w_{(m+1,n)} - 3w_{(m+2,n)} + w_{(m+3,n)}.$$

Assuming that $m \geqslant 3$, $n \geqslant 0$, and using expression (5.1) for elements of w^Δ, one has after some manipulation that

$$(g_3^* * w^\Delta)_{(-m,-n)} = \frac{1}{2^{m+n+1}}\binom{m+n}{m}\frac{(m-n)^3 - 3(m^2 - n^2) + 2(m-n)}{(m+n-2)(m+n-1)(m+n)}.$$

Let $N = m + n > 3$. Then $m = N - n$, and so

$$|(g_3^* * w^\Delta)_{(-m,-n)}| \leqslant \frac{1}{2^{N+1}}\binom{N}{n}\frac{|N-2n|^3 + (3N+2)|N-2n|}{(N-2)(N-1)N}.$$

Suppose $X_1, X_2, \dots$ are independent random variables with an identical distribution $\mathbb{P}(X_i = \pm 1) = \frac{1}{2}$. Then, by a well known probabilistic result, the so-called Khintchine inequality [**11**], for any $p > 0$ there exists a constant c_p such that

$$\mathbb{E}\Bigl|\sum_{i=1}^{N} X_i\Bigr|^p = \frac{1}{2^N}\sum_{n=0}^{N}\binom{N}{n}|N-2n|^p \leqslant c_p N^{\frac{p}{2}} \quad \text{for all } N.$$

Thus for some $C > 0$ and all sufficiently large N

$$\sum_{\substack{m\geqslant 3, n\geqslant 0\\ m+n=N}} |(g_3^* * w^\Delta)_{(-m,-n)}| \leqslant \frac{C}{N^{3/2}}.$$

This, together with the observation that the boundary terms $w^\Delta_{(-m,-n)}$ with $m \leqslant 3$ are exponentially small in $N = m + n$, proves that $g_3^* * w^\Delta \in \ell^1(\mathbb{Z}^d, \mathbb{R})$.

Similarly one shows that other third powers $(1-u)^2(1-v)$, $(1-u)(1-v)^2$, $(1-v)^3$ belong to $\mathfrak{m}_f$ as well. Moreover, $u - 1 \equiv -(v-1) \bmod \langle f\rangle$ and $\mathfrak{m}_f \supset \langle f\rangle$. Therefore from $(u-1)^2 \notin \mathfrak{m}_f$ we conclude that $(u-1)(v-1) \notin \mathfrak{m}_f$ and $(v-1)^2 \notin \mathfrak{m}_f$. Thus we have exactly identified the multiplier ideal of $f(u, v) = 2 - u - v$ to be $\mathfrak{m}_f = \langle f\rangle + \mathfrak{I}_2^3$.

6. Symbolic covers

For every nonzero summable homoclinic point $z \in \Delta^1_\alpha(X)$ we construct here a shift-equivariant group homomorphism from $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ to X. Indeed this map is surjective when restricted to a ball of finite radius in $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, and so provides a symbolic cover of X.

According to Proposition 5.2, every homoclinic point $z \in \Delta^1_\alpha(X)$ has the form $z = \rho(w^{(g)})$ for some $g \in \mathfrak{m}_f$, where $w^{(g)} \in \ell^1(\mathbb{Z}^d, \mathbb{R})$. We define group homomorphisms $\widetilde{\xi}_g\colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \to \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ and $\xi_g\colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \to \mathbb{T}^{\mathbb{Z}^d}$ by

$$\widetilde{\xi}_g(v) = {w^{(g)}}^*(\tilde{\sigma})(v) = w^{(g)} * v \quad\text{and}\quad \xi_g(w) = \rho(\widetilde{\xi}_g(w)).$$

These maps are well-defined since $w^{(g)} \in \ell^1(\mathbb{Z}^d, \mathbb{R})$, and commute with the appropriate $\mathbb{Z}^d$-actions.

Proposition 6.1. *For every $g \in \mathfrak{m}_f$,*

$$\xi_g(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})) = \begin{cases} \{0\} & \text{if } g \in \langle f\rangle. \\ X & \text{if } g \in \mathfrak{m}_f \smallsetminus \langle f\rangle. \end{cases} \tag{6.1}$$

We first establish two lemmas.

Lemma 6.2. *For every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ and $g \in \mathfrak{m}_f$,*

$$(f(\tilde{\sigma}) \circ \widetilde{\xi}_g)(v) = f^* * w^{(g)} * v = g^* * v = g(\tilde{\sigma})(v). \tag{6.2}$$

Proof. The proof of Proposition 5.2 shows, after taking Fourier transforms, that (6.2) holds whenever $g \in \mathfrak{m}_f$ and $v \in \ell^1(\mathbb{Z}^d, \mathbb{R})$.

For $K \geqslant 1$ put $V_K = \{v \in \ell^\infty(\mathbb{Z}^d, \mathbb{R}) : \|v\|_\infty \leqslant K\}$. Then V_K is shift-invariant and compact in the topology of pointwise convergence, and the set $V_K^1 = \mathsf{V}_K \cap \ell^1(\mathbb{Z}^d, \mathbb{R})$ is dense in V_K. For $v \in V_K^1$ clearly $\widetilde{\xi}_g(v) = w^{(g)} * v$ and $(f(\tilde{\sigma}) \circ \widetilde{\xi}_g)(v) = g^* * v$. Since $\widetilde{\xi}_g$ and $f(\tilde{\sigma})$ are continuous on V_K, these equations continue to hold for all $v \in V_K$. Letting $K \to \infty$ shows that (6.2) holds for all $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ and $g \in \mathfrak{m}_f$.

For the last assertion, recall that for every $v \in R_d$,

$$f(\tilde{\sigma})(\widetilde{\xi}_g(v)) = f^* * w^{(g)} * v = g^* * v \in R_d \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z}).$$

Hence $\xi_g(v) = \rho(\widetilde{\xi}_g(v)) \in X$ for every $v \in R_d$. The continuity argument above then shows that $\xi_g(v) \in X$ for every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$. □

Lemma 6.3. *Let $g \in \mathfrak{m}_f \smallsetminus \langle f \rangle$, and put $K = \sum_{\mathbf{n} \in \mathbb{Z}^d} |f_{\mathbf{n}}|$. Then $\xi_g(V_K) = X$, and so $\xi_g(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})) = X$. Furthermore, the restriction of ξ_g to V_K, or to any other bounded, closed, shift-invariant subspace of $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, is continuous in the product topology.*

Proof. Let $x \in X$. Choose $w \in W_f$ with $\rho(w) = x$ and $0 \leqslant w_{\mathbf{n}} < 1$ for all $\mathbf{n} \in \mathbb{Z}^d$. If $v = f(\tilde{\sigma})(w)$, then $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ and $-K \leqslant v_{\mathbf{n}} \leqslant K$ for every $\mathbf{n} \in \mathbb{Z}^d$, so that $v \in V_K$.

Since $\widetilde{\xi}_g$ commutes with $f(\tilde{\sigma})$, we see that $\xi_g(v) = \rho(\widetilde{\xi}_g(v)) = g(\alpha)(x)$. This shows that $g(\alpha)(X) \subset \xi_g(V_K) \subset X$.

We claim that $g(\alpha)(X) = X$. For $h + \langle f \rangle \in R_d/\langle f \rangle$ annihilates $g(\alpha)X$ iff $gh + \langle f \rangle$ annihilates X iff $gh \in \langle f \rangle$ iff $h \in \langle f \rangle$, since f is irreducible and $g \notin \langle f \rangle$. This shows that $g(\alpha)(X)$ and X have the same annihilator, and so $g(\alpha)(X) = X$.

Continuity of ξ_g follows as in the previous lemma. □

Proof of Proposition 6.1. If $g = h \cdot f \in \langle f \rangle$ for some $h \in R_d$, then $w^{(g)} = h^* \in R_d$, and hence $\widetilde{\xi}_g(v) = h * v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ for every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, showing that $\xi_g(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})) = \{0\}$. The case $g \in \mathfrak{m}_f \smallsetminus \langle f \rangle$ is handled by Lemma 6.3. □

7. Proof of Theorem 1.2

We use the fact that entropy equals the growth rate of separated sets, and that by using homoclinic points we can approximate elements in such sets with periodic points.

Lemma 7.1. *Let $\{\Gamma_n\}_{n \geqslant 1}$ be a sequence of finite-index subgroups of $\mathbb{Z}^d$ with $\langle \Gamma_n \rangle \to \infty$ as $n \to \infty$. Then there exists a sequence $\{Q_n\}_{n \geqslant 1}$ of subsets of $\mathbb{Z}^d$ such that*

(1) *Each Q_n is a fundamental domain for Γ_n, i.e. the collection $\{Q_n + \mathbf{m} : \mathbf{m} \in G_n\}$ is disjoint and has union $\mathbb{Z}^d$; and*
(2) *$\{Q_n\}_{n \geqslant 1}$ is a Følner sequence in $\mathbb{Z}^d$.*

Proof. This is an easily proved special case of [**3**, Cor. 5.6]. □

Definition 7.2. Let $Q \subset \mathbb{Z}^d$ and $\varepsilon > 0$. We say that $E \subset X$ is (Q, ε)-*spanning* in X if, for every $x \in X$ there is a $y \in E$ such that $|x_{\mathbf{n}} - y_{\mathbf{n}}| < \varepsilon$ for every $\mathbf{n} \in Q$. Dually, $F \subset X$ is (Q, ε)-*separated* in X if, for every distinct pair x, y of points in F, there is an $\mathbf{n} \in Q$ with $|x_{\mathbf{n}} - y_{\mathbf{n}}| \geqslant \varepsilon$.

Lemma 7.3. *For every $\varepsilon > 0$ there exists a finite set A_ε with the following property: if Γ is a finite-index subgroup of $\mathbb{Z}^d$ and Q is a fundamental domain for Γ, then $\mathsf{Fix}_\Gamma(\alpha)$ is $\big(\bigcap_{\mathbf{m} \in A_\varepsilon}(Q - \mathbf{m}), \varepsilon\big)$-spanning in X.*

Proof. Fix $g \in \mathfrak{m}_f \smallsetminus \langle f \rangle$, and define $w^{(g)} \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ as in (5.2). Let $\varepsilon > 0$, and put $K = \sum_{\mathbf{n} \in \mathbb{Z}^d} |f_{\mathbf{n}}|$. Choose a finite subset A_ε of $\mathbb{Z}^d$ so that $\sum_{\mathbf{n} \in \mathbb{Z}^d \smallsetminus A_\varepsilon} |w_{\mathbf{n}}^{(g)}| < \varepsilon/K$.

Since $\widetilde{\xi}_g(V_K) = X$ by Proposition 6.1, for every $x \in X$ there is a $v \in V_K$ with $\xi_g(v) = x$. Define $v' \in V_\Gamma(\mathbb{Z})$ by requiring that $v'_{\mathbf{n}} = v_{\mathbf{n}}$ for every $\mathbf{n} \in Q$, and

extending v' by Γ-periodicity. Our choice of A_ε implies that $|\widetilde{\xi}_g(v)_{\mathbf{n}} - \widetilde{\xi}_g(v')_{\mathbf{n}}| < \varepsilon$ for every $\mathbf{n} \in \bigcap_{\mathbf{m}\in A_\varepsilon}(Q-\mathbf{m})$. Let $x' = \rho(v')$. Then $x \in \mathsf{Fix}_\Gamma(\alpha)$ and $|x_{\mathbf{n}} - x'_{\mathbf{n}}| < \varepsilon$ for every $\mathbf{n} \in \bigcap_{\mathbf{m}\in A_\varepsilon}(Q-\mathbf{m})$. □

We write

$$\Omega(f) = \{\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in \mathsf{U}(f) : \text{ each } \omega_j \text{ is a root of unity}\}$$

for the set of torsion points in $\mathsf{U}(f)$. If $\Omega(f) \neq \varnothing$, set

$$\Gamma(f) = \{\mathbf{n} \in \mathbb{Z}^d : \boldsymbol{\omega}^{\mathbf{n}} = 1 : \text{ for every } \boldsymbol{\omega} \in \Omega(f)\}.$$

Then $\Gamma(f) \in \mathcal{F}$, and we can find $N(f) > 0$ with $\Gamma(f) \subset N(f)\cdot\mathbb{Z}^d$.

Lemma 7.4. *Let Γ be a finite-index subgroup, and put $\Omega_\Gamma(f) = \Omega(f) \cap \Omega_\Gamma$. Then $\mathsf{Fix}_\Gamma(\alpha)$ is finite if and only if $\Omega_\Gamma(f) = \varnothing$. If $\Omega_\Gamma(f) \neq \varnothing$, then $\mathsf{Fix}^\circ_\Gamma(\alpha) \cong \mathbb{T}^{|\Omega_\Gamma(f)|}$ and $\mathsf{Fix}^\circ_\Gamma(\alpha) \subset \mathsf{Fix}_{N(f)\cdot\mathbb{Z}^d}(\alpha)$.*

Proof. We denote by $W_\Gamma(\mathbb{C}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{C})$ the linear span of $\{v^{(\boldsymbol{\omega})} : \omega \in \Omega_\Gamma(f)\}$, where $v^{(\boldsymbol{\omega})}$ is defined in (2.1). Write $V_\Gamma(\mathbb{R}) = W_\Gamma(\mathbb{C}) \cap \ell^\infty(\mathbb{Z}^d, \mathbb{R}) \subset W_f$ for the real part of $W_\Gamma(\mathbb{C})$. The dimension of $V_\Gamma(\mathbb{R})$ equals $|\Omega_\Gamma(f)|$, and $\mathsf{Fix}^\circ_\Gamma(\alpha) = \rho(V_\Gamma(\mathbb{R})) \cong \mathbb{T}^{|\Omega_\Gamma(f)|}$. Since $V_\Gamma(\mathbb{R}) \subset V_{N(f)\cdot\mathbb{Z}^d}(\mathbb{R})$, applying ρ shows that $\mathsf{Fix}^\circ_\Gamma(\alpha) \subset \mathsf{Fix}_{N(f)\cdot\mathbb{Z}^d}(\alpha)$. □

Lemma 7.5. *For every $\varepsilon > 0$ there is an $M(\varepsilon) > 0$ with the following property: for each $\Gamma \in \mathcal{F}$, every $(\mathbb{Z}^d, \varepsilon)$-separated set in $\mathsf{Fix}^\circ_\Gamma(\alpha)$ has cardinality $< M(\varepsilon)$.*

Proof. By Lemma 7.4, for every $\Gamma \in \mathsf{Fix}_\Gamma$ we have that $\mathsf{Fix}^\circ_\Gamma(\alpha)$ is a subtorus of the fixed finite-dimensional torus $\mathsf{Fix}_{N(f)\cdot\mathbb{Z}^d}(\sigma)$. If $Q = \{0, \dots, N(f)-1\}^d$, there is an $M(\varepsilon) > 0$ such that every (Q, ε)-separated set in $\mathsf{Fix}_{N(f)\cdot\mathbb{Z}^d}(\sigma)$ has cardinality $< M(\varepsilon)$. By periodicity, every $(\mathbb{Z}^d, \varepsilon)$-separated set in $\mathsf{Fix}_{N(f)\cdot\mathbb{Z}^d}(\sigma)$ (and hence in $\mathsf{Fix}^\circ_\Gamma(\alpha)$) has cardinality $< M(\varepsilon)$. □

Lemma 7.6. *Let $Q \subset \mathbb{Z}^d$ and Γ be a finite-index subgroup of $\mathbb{Z}^d$. Suppose that $\varepsilon > 0$ and that $F \subset \mathsf{Fix}_\Gamma(\alpha)$ is a (Q, ε)-separated subset with cardinality L. Then F intersects at least $L/M(\varepsilon)$ distinct cosets of $\mathsf{Fix}^\circ_\Gamma(\alpha)$ in $\mathsf{Fix}_\Gamma(\alpha)$, where $M(\varepsilon)$ is given in Lemma 7.5.*

Proof. This is an immediate consequence of Lemma 7.5. □

Proof of Theorem 1.2. For a finite subset $Q \in \mathbb{Z}^d$, let $r_Q(\varepsilon)$ denote the largest cardinality of a (Q, ε)-separated set in X. According to [**2**, Prop. 2.1], for every Følner sequence $\{L_n\}_{n\geqslant 1}$ in $\mathbb{Z}^d$, we have that

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{|L_n|} \log r_{L_n}(\varepsilon) = \mathsf{h}(\alpha). \tag{7.1}$$

Let $\{\Gamma_n\}_{n\geqslant 1}$ be a sequence in $\mathcal{F}$ with $\langle \Gamma_n \rangle \to \infty$ as $n \to \infty$. By Lemma 7.1, there exists a Følner sequence $\{Q_n\}_{n\geqslant 1}$ of fundamental domains for the Γ_n.

Fix $\varepsilon > 0$ and use Lemma 7.3 to find a finite set $A_{\varepsilon/3} \subset \mathbb{Z}^d$ such that $\mathsf{Fix}_{\Gamma_n}(\alpha)$ is $(Q'_n, \varepsilon/3)$-spanning in X for every $n \geqslant 1$, where $Q'_n = \bigcap_{\mathbf{m}\in A_{\varepsilon/3}}(Q_n - \mathbf{m})$. Note that $\{Q'_n\}_{n\geqslant 1}$ is again a Følner sequence in $\mathbb{Z}^d$ with $|Q'_n|/|Q_n| \to 1$ as $n \to \infty$. We may assume $Q'_n \neq \varnothing$ for all $n \geqslant 1$.

For all $n \geqslant 1$ choose a maximal (Q'_n, ε)-separated set $F_n \subset X$ with cardinality $r_{Q'_n}(\varepsilon)$. We fix n for the moment and choose for every $y \in F_n$ an element $z(y) \in \mathsf{Fix}_{\Gamma_n}(\alpha)$ with $|y_{\mathbf{n}} - z(y)_{\mathbf{n}}| < \varepsilon/3$ for all $\mathbf{n} \in Q'_n$. The points $z(y)$ must be distinct

for different y, so $F'_n = \{z(y) : y \in F_n\}$ also has cardinality $|F_n|$. Lemma 7.6 shows that there is an $M(\varepsilon) > 0$ (which depends on ε but not on n) such that F'_n intersects at least $|F'_n|/M(\varepsilon/3)$ distinct cosets of $\mathsf{Fix}^{\circ}_{\Gamma_n}(\alpha)$ in $\mathsf{Fix}_{\Gamma_n}(\alpha)$. Hence

$$\mathsf{P}_{\Gamma_n}(\alpha) = |\,\mathsf{Fix}_{\Gamma_n}(\alpha)/\,\mathsf{Fix}^{\circ}_{\Gamma_n}(\alpha)| \geqslant \frac{|F'_n|}{M(\varepsilon/3)} = \frac{1}{M(\varepsilon/3)} r_{Q'_n}(\varepsilon)$$

for every $n \geqslant 1$. It follows that

$$\liminf_{n\to\infty} \frac{1}{|\mathbb{Z}^d/\Gamma_n|} \log \mathsf{P}_{\Gamma_n}(\alpha) \geqslant \liminf_{n\to\infty} \frac{1}{|Q_n|} \log r_{Q'_n}(\varepsilon) = \liminf_{n\to\infty} \frac{1}{|Q'_n|} \log r_{Q'_n}(\varepsilon).$$

Letting $\varepsilon \to 0$, invoking (7.1), and combining with Theorem 1.1 completes the proof. □

8. Specification

Specification is a strong orbit tracing property that has many uses. Ruelle [**12**] investigated the extension of this notion to topological $\mathbb{Z}^d$-actions. In [**8**] it was shown that expansive algebraic $\mathbb{Z}^d$-actions with completely positive entropy satisfy several flavors of specification. The proof made crucial use of the existence of summable homoclinic points. By Proposition 5.2, this tool remains available for the (nonexpansive) actions $\alpha_{R_d/\langle f\rangle}$ when $\mathsf{U}(f)$ is finite. In this section we extend previous results to such actions.

Definition 8.1. Let β be a $\mathbb{Z}^d$-action by homeomorphisms of a compact metric space (X,ρ).

(1) The action β has *strong specification* if there exists, for every $\varepsilon > 0$, a number $p(\varepsilon) > 0$ with the following property: for every finite collection $\{Q_1,\dots,Q_t\}$ of finite subsets of $\mathbb{Z}^d$ with

$$\operatorname{dist}(Q_j,Q_k) := \min_{\mathbf{m}\in Q_j,\ \mathbf{n}\in Q_k} \|\mathbf{m}-\mathbf{n}\| \geqslant p(\varepsilon) \quad (1 \leqslant j < k \leqslant t), \tag{8.1}$$

every collection $\{x^{(1)},\dots,x^{(t)}\} \subset X$, and every $\Gamma \in \mathcal{F}$ with

$$\operatorname{dist}(Q_j+\mathbf{k},Q_k) \geqslant p(\varepsilon) \quad (1 \leqslant j < k \leqslant t,\ \mathbf{k} \in \Gamma \smallsetminus \{\mathbf{0}\}),$$

there is a $y \in \mathsf{Fix}_\Gamma(\beta)$ with

$$\rho(\beta^{\mathbf{n}}y, \beta^{\mathbf{n}}x^{(j)}) < \varepsilon \text{ for all } \mathbf{n} \in Q_j,\ (1 \leqslant j \leqslant t). \tag{8.2}$$

(2) The action β has *homoclinic specification* if, for every $\varepsilon > 0$, there is a $p(\varepsilon) > 0$ such that for every finite collection $\{Q_1,\dots,Q_t\}$ of finite subsets of $\mathbb{Z}^d$ satisfying (8.1) and every $\{x^{(1)},\dots,x^{(t)}\} \subset X$, there is a $y \in \Delta_\beta(X)$ satisfying (8.2).

Theorem 8.2. *Let $d \geqslant 2$ and $f \in R_d$ have finite $\mathsf{U}(f)$. Then $\alpha_{R_d/\langle f\rangle}$ has both strong specification and homoclinic specification.*

Proof. Let $\alpha = \alpha_{R_d/\langle f\rangle}$ and $X = X_{R_d/\langle f\rangle}$. Using the notation of Proposition 5.2, choose $g \in \mathfrak{m}_f \smallsetminus \langle f\rangle$, with corresponding $x^{(g)} \in \Delta^1_\alpha$. By the proof of Lemma 6.3, we can find $y^{(j)} \in X$ with $g(\alpha)(y^{(j)}) = x^{(j)}$ for $1 \leqslant j \leqslant t$. The proof of [**8**, Thm. 5.2], applied to the $y^{(j)}$ and replacing the fundamental homoclinic point with $x^{(g)}$, now yields the required y. □

Remark 8.3. For $d = 1$ and $\mathsf{U}(f) \neq \varnothing$, both strong specification and homoclinic specification always fail, although a weaker form still holds [**6**]. This again illustrates the difference between $d = 1$ and $d \geqslant 2$.

9. Further remarks

An alternative approach to proving Theorem 1.2 uses Gelfond's deep results on algebraic numbers (see [**4**, p. 28]). Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)$ have algebraic coordinates, and recall that these are called *multiplicatively independent* if the only $\mathbf{n} \in \mathbb{Z}^d$ for which $\boldsymbol{\xi}^{\mathbf{n}} := \xi_1^{n_1} \cdots \xi_d^{n_d} = 1$ is $\mathbf{n} = \mathbf{0}$.

Theorem 9.1 (Gelfond, [**4**, Thm. III])**.** *Suppose that $\boldsymbol{\xi} = (\xi_1 \dots, \xi_d)$ has algebraic coordinates that are multiplicatively independent. Then for every $\varepsilon > 0$ there are only finitely many $\mathbf{n} \in \mathbb{Z}^d$ for which $|\boldsymbol{\xi}^{\mathbf{n}} - 1| < e^{-\varepsilon \|\mathbf{n}\|}$, where $\|\mathbf{n}\| = \max\{|n_1|, \dots, |n_d|\}$.*

Let $f \in R_d$ have finite $\mathsf{U}(f)$. Define $\log_0 t$ for $t \geqslant 0$ to be $\log t$ if $t > 0$ and 0 if $t = 0$. According to Lemma 2.1, for each $\Gamma \in \mathcal{F}$,

$$\frac{1}{|\Omega_\Gamma|} \log \mathsf{P}_\Gamma(\alpha_{R_d/\langle f \rangle}) = \frac{1}{|\Omega_\Gamma|} \sum_{\boldsymbol{\omega} \in \Omega_\Gamma} \log_0 |f(\boldsymbol{\omega})|. \tag{9.1}$$

Now $\log|f|$ has only finitely many logarithmic singularities, and by Proposition 4.2 these all have algebraic coordinates. We can therefore use Gelfond's result to control the few potentially large negative values of $\log_0 |f|$ for $\boldsymbol{\omega} \in \Omega_\Gamma$ near one of these singularities, to show that the Riemann sums in (9.1) will converge to the limit $\mathsf{m}(f) = \mathsf{h}(\alpha_{R_d/\langle f \rangle})$.

To make a similar argument work when $\mathsf{U}(f)$ is infinite, we would need an estimate of the form $\operatorname{dist}(\mathsf{U}(f), \boldsymbol{\omega}) \geqslant e^{-\varepsilon \cdot o(\boldsymbol{\omega})}$, where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \notin \mathsf{U}(f)$ has coordinates which are roots of unity, and $o(\boldsymbol{\omega})$ denotes the order of $\boldsymbol{\omega}$ in $\mathbb{S}^d$. Such estimates, however, do not appear to be available.

We remark that if we replace averages of $\log|f|$ over the finite subgroups Ω_Γ by averages over a sequence $\{K_n\}$ of compact connected subgroups in $\mathbb{S}^d$ that become uniformly distributed, then a result of Lawton [**5**] shows that for every nonzero $f \in R_d$ we do have convergence,

$$\int_{K_n} \log|f| \, d\lambda_{K_n} \to \int_{\mathbb{S}^d} \log|f| \, d\lambda_{\mathbb{S}^d} \text{ as } n \to \infty,$$

and so the diophantine issues disappear.

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